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ON THE DIFFRACTION OF A SURFACE GRAVITATIONAL WAVE OVER A SMALL IRREGULARITY OF THE BOTTOM

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S. F. DOTSENKO and L. V. CHERKESOV
(Sevastopol')

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The plane and three-dimensional problems of diffraction of a plane gravitational wave over a tank bottom irregularity of arbitrary shape is considered using the general linear theory of irrotational waves in a fluid of finite depth.

Such investigations using the theory of long waves were carried out in [1—3] by numerical methods which in a general linear formulation can only be achieved in the case of obstacles of the simplest shape [4—6]. In this paper the height of the bottom irregularity is assumed small in comparison with the tank depth. This makes it possible to obtain asymptotic expressions for scattered waves, and to analyze these for a wide class of shapes of bottom irregularities.

1. Let a perfect incompressible homogeneous heavy fluid occupy the region $-\infty < x, y < +\infty$, $-H(x, y) \leqslant z \leqslant 0$, where x and y are the horizontal and z the vertical coordinates, $H = H_0 - h(x, y)$ and $H_0 = \text{const}$, $h \to 0$ when $R = \sqrt{x^2 + y^2} \to \infty$. We shall analyze in linear formulation the diffraction of a plane progressing wave propagating from $x = -\infty$ over the bottom irregularity defined by function h, assuming the fluid motion to be irrotational and parameter $\varepsilon = h_0 H_0^{-1}$ small $(h_0 = \max |h|)$.

We specify the velocity potential φ_0 and the free surface rise ζ_0 of the insident wave by

$$\varphi_0 = \operatorname{Re}\{\Phi_0 \exp(-i\sigma t)\}, \quad \zeta_0 = \operatorname{Re}\{A_0 \exp[i(r_0 x - \sigma t)]\} \qquad (1.1)$$
$$\Phi_0 = -iA_0 g\sigma^{-1} \operatorname{ch} r_0 (z + H_0) \operatorname{ch}^{-1} r_0 H \exp(ir_0 x)$$
$$\sigma = (gr_0 \operatorname{th} r_0 H_0)^{1/2}$$

where A_0 and r_0 are known quantities and g is the free-fall acceleration.

We denote by φ and ζ the velocity potential and the rise of the diffracted wave free surface, respectively. We set

$$\varphi = \varphi_0 + \operatorname{Re}\{\Phi \exp(-i\sigma t)\}, \quad \zeta = \zeta_0 + \operatorname{Re}\{A \exp(-i\sigma t)\}$$

and introduce dimensionless variables by formulas

$$\{x, y, z\} = H_0\{x', y', z'\}, \quad t = \sigma^{-1} t'$$
$$\{\Phi_0, \Phi\} = A_0 g\sigma^{-1}\{\Phi_0', \varepsilon\Phi'\}$$
$$\{\zeta_0, A\} = A_0\{\zeta_0', \varepsilon A'\}, \quad h = h_0 h', \quad r_0 = H_0^{-1} r_0'$$
$$\sigma = g^{1/2} H_0^{-1/2} \sigma'$$

For the determination of potential Φ' in dimensionless variables (at which primes are subsequently omitted) we have the problem

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \quad \Phi_z - \sigma^2\Phi = 0 \quad (z = 0) \tag{1.2}$$

$$\Phi_{0z} + \varepsilon(\Phi_z - h_x\Phi_{0x} - h_y\Phi_{0y}) - \varepsilon^2(h_x\Phi_x + h_y\Phi_y) = 0 \tag{1.3}$$
$$(z = -1 + \varepsilon h)$$

$$\Phi_0 = -i \operatorname{ch} r_0(z+1)\operatorname{ch}^{-1} r_0 \exp(ir_0x), \quad \sigma = \sqrt{r_0 \operatorname{th} r_0}$$

The condition of radiation which implies that the scattered wave potential Φ exp $(-i\sigma t)$ defines waves propagating from the bottom irregularity to infinity, must also be satisfied. Function $A(x, y)$ is defined by formula $A = i\Phi(x, y, 0)$ which follows from the Cauchy–Lagrange integral.

The boundary condition (1.3) creates fundamental mathematical difficulties in the derivation of an exact solution of the problem. We shall seek a solution of problem (1.2), (1.3) of form $\Phi = \Phi_1 + \varepsilon\Phi_2 + \varepsilon^2\Phi_3 + \ldots$ using the small parameter ε [7]. Similarly $A = A_1 + \varepsilon A_2 + \varepsilon^2 A_3 + \ldots$. Then $A_1 = i\Phi_1(x, y, 0)$ and Φ_1 satisfies Eq. (1.2) and the boundary condition

$$\Phi_{1z} = h_x\Phi_{0x} + h_y\Phi_{0y} - h\Phi_{0zz} \quad (z = -1) \tag{1.4}$$

A recurrent sequence of boundary value problems can be derived for the determination of Φ_n and A_n $(n \geqslant 2)$. Below we restrict the analysis of wave diffraction to the first approximation solution . For brevity we omit subscripts at the unkown Φ_1 and A_1.

2. The integral representation for Φ is obtained from (1.2) and (1.4) using the Fourier transform in x, y. In polar coordinates

$$\Phi = (2\pi)^{-2}\int_c [\Delta \operatorname{sh} r(z+1) + (\sigma^2 \operatorname{th} r - r)\operatorname{ch} r(z+1)](r\Delta)^{-1} I\, dr \tag{2.1}$$

$$I = \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\exp[irR\cos(\theta - \gamma)]\, d\theta \tag{2.2}$$

$$f(m, n) = ir_0 m F(m - r_0, n)\operatorname{ch}^{-1} r_0, \quad \Delta(r) = r \operatorname{th} r - \sigma^2$$

$$\{x, y\} = R\{\cos\gamma, \sin\gamma\}, \quad \{m, n\} = r\{\cos\theta, \sin\theta\}$$

where $F(m, n)$ is the Fourier transform of function $h(x, y)$ and c is the integration path in the complex r-plane moving along the ray $\operatorname{Re} r \geqslant 0$ and bypassing point $r = r_0$ $(\Delta(r_0) = 0)$ from below along a small semicircle.

Let us consider the asymptotic behavior of Φ and A as $R \to \infty$. For this we apply to integral (2.2) the method of stationary phase, then substitute the principal term of the asymptotics into (2.1) and investigate the integral thus obtained, using the theory of residues. We finally obtain

$$\operatorname{Re}\{\Phi\exp(-i\sigma t)\} = R^{-1/2} \operatorname{ch} r_0(z+1)\operatorname{ch}^{-1} r_0 B \times \tag{2.3}$$
$$\operatorname{Re}\{\Pi(\gamma)\exp[i(r_0R - \sigma t - \pi/4)]\} + O(R^{-1})$$

$$\operatorname{Re}\{A\exp(-i\sigma t)\} = R^{-1/2}B\operatorname{Re}\{\Pi(\gamma)\exp[i(r_0R-\sigma t+\pi/4)]\}+O(R^{-1}) \tag{2.4}$$

$$B = (2\pi)^{-1/2}r_0^{1/2}(r_0^2-\sigma^4)(r_0^2+\sigma^2-\sigma^4)^{-1}$$

$$\Pi = \cos\gamma\, F(r_0(\cos\gamma-1),\ r_0\sin\gamma)|$$

If the incident wave propagates at angle γ_0 to the positive x-axis, scattered waves are also defined by formulas (2.3) and (2.4) but with the multiplier Π replaced by

$$\Pi_1 = \cos(\gamma-\gamma_0)F(m_1, n_1),\quad m_1 = r_0(\cos\gamma-\cos\gamma_0)$$

$$n_1 = r_0(\sin\gamma-\sin\gamma_0)$$

The dependence of the scattered wave amplitude on γ is defined by function Π_1 which is determined by the two-dimensional transform of Fourier function h. Function Π_1 is simplified in the following cases;

$$h = h_3([\alpha x^2+\beta y^2]^{1/2}) \tag{2.5}$$

$$\Pi_1 = \frac{2\pi}{\alpha\beta}\cos(\gamma-\gamma_0)\int_0^\infty Rh_3(R)J_0(m_2R)\,dR,\quad m_2 = \left(\frac{m_1^2}{\alpha}+\frac{n_1^2}{\beta}\right)^{1/2}$$

$$h = h_1(x)h_2(y),\quad \Pi_1 = \cos(\gamma-\gamma_0)F_1(m_1)F_2(n_1) \tag{2.6}$$

where $F_s(m)$, $s = 1, 2$ are one-dimensional Fourier transforms of functions h_s and $J_0(z)$ is a Bessel function of the first kind. The case of $\alpha = \beta$ corresponds to an axisymmetric irregularity of the bottom.

3. Let us consider the long- and short-wave asymptotics of the distant field of scattered waves. We assume that function h depends on parameter $\delta > 0$, that $h = h(x/\delta, y/\delta)$, and that the wave propagates at angle γ_0 to the x-axis. Then, denoting by G the set of points (ξ, η) for which $h(\xi, \eta) \neq 0$, we obtain

$$\Pi_1 = \delta^2\cos(\gamma-\gamma_0)\Pi_2(\nu),\quad \nu = \delta r_0 \tag{3.1}$$

$$\Pi_2 = \iint_G h(\xi,\eta)\exp[-i\nu\Psi(\xi,\eta)]\,d\xi\,d\eta \tag{3.2}$$

$$\Psi = (\cos\gamma-\cos\gamma_0)\xi+(\sin\gamma-\sin\gamma_0)\eta$$

The long wave approximation $(\nu\to 0)$. Expanding function Π_2 in power series in parameter ν, we obtain

$$\Pi_2 = \Lambda_0 - i\nu[(\cos\gamma-\cos\gamma_0)\Lambda_1+(\sin\gamma-\sin\gamma_0)\Lambda_2]+O(\nu^2) \tag{3.3}$$

$$\Lambda_0 = \iint_G h(\xi,\eta)\,d\xi\,d\eta,\quad \Lambda_1 = \iint_G \xi h(\xi,\eta)\,d\xi\,d\eta$$

$$\Lambda_2 = \iint_G \eta h(\xi,\eta)\,d\xi\,d\eta$$

Formulas (3.1) and (3.3) imply that when $\Lambda_0 \neq 0$ and $\nu\to 0$ the dependence of the amplitude of the scattered wave free surface rise on γ is close to that defined by function $|\cos(\gamma-\gamma_0)|$. Hence in a system of coordinates whose x-axis coincides with the ray $\gamma = \gamma_0$ the distant wave field weakly depends on γ_0 and on the shape of the bottom irregularity when $\delta^2\Lambda_0 = \text{const} \neq 0$. If $\Lambda_0 = 0$, $\Lambda_1^2 + \Lambda_2^2 > 0$, the distribution of wave amplitudes with respect to γ is determined by the constants $\Lambda_{1,2}$.

Formulas (2.3), (2.4), (3.1), and (3.3) make possible the derivation for a scattered wave of the eqution of the line of constant amplitude of the free surface rise, and of velocity vector components along the x, y, and z axes (with $z = \text{const}$). In polar coordinates these equations are, respectively, of the form

$$R = c\cos^2\gamma_1, \quad R = c\cos^4\gamma_1, \quad R = c\sin^2 2\gamma_1, \quad R = c\cos^2\gamma_1$$

$$(\gamma_1 = \gamma - \gamma_0,\ c = \text{const})$$

Short-wave asymptotics $\nu \to \infty$. Let us consider the asymptotic behavior of Π_1 as $(\nu \to \infty)$. If $\gamma = \gamma_0$, Π_2 is independent of ν $(\Pi_2 = \Lambda_0)$. Assuming that $\gamma \neq \gamma_0$ we apply to integral (3.2) the method of stationary phase for multiple integrals [8]. Since $|\Delta\Psi(\xi, \eta)| \neq 0$, the integral Π_2 has no internal stationary points, and for a fairly smooth function h and $G = (-\infty, \infty)\times(-\infty, \infty)$ that integral and, consequently, the scattered wave amplitude are of a fairly high order of smallness with respect to parameter ν. In particular $\Pi_2 = O(\nu^{-\infty})$ if $h \in C^\infty$.

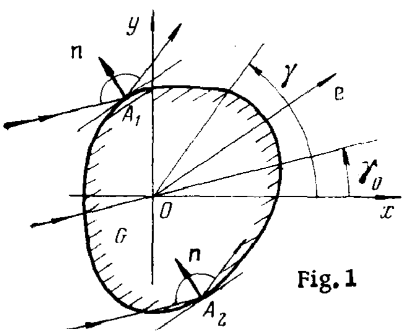


Fig. 1

If the region G is bounded and function h fairly smooth, the basic contribution to the asymptotics of integral Π_2 is provided by the boundary points of that region [9]. It can be shown by integrating (3.2) by parts (formula (1.3) in [9]) that the higher the smoothness of h in the neighborhood of the region boundary the lower is the scattered wave amplitude in the direction $\gamma \neq \gamma_0$. Thus, when conditions $h = \partial h / \partial x = \ldots = \partial^n h / \partial x^n = 0\ (\gamma \neq \pm\gamma_0)$ or $h = \partial h / \partial y = \ldots = \partial^n h / \partial y^n = 0\ (\gamma \neq \gamma_0, \pi - \gamma_0)$ are satisfied at the boundary of G, then $\Pi_2 = O(\nu^{-n-1})$.

Let us consider an irregularity of the bottom with a vertical side boundary ($h \neq 0$ at the boundary of G). As $\nu \to \infty$, the basic contribution to the asymptotics of Π_2 is provided by the boundary points at which the unit vector of the normal to the boundary is parallel to vector $\nabla\Psi$ [9], where $\nabla\Psi = (\cos\gamma_0 - \cos\gamma, \sin\gamma_0 - \sin\gamma)$. At such points ($A_1$ and A_2 in Fig. 1) the wave reflection angle is equal to its incident angle, with the reflected ray at angle γ to the x-axis. It can be shown, as in [9], that in the case of a smooth boundary the contribution of points A_s to the asymptotics of Π_2 is equal $O(\nu^{-(n_s+1)/n_s})$, where $n_s \geqslant 2$ is the order of contact of the tangent and the boundary G at point A_s. If the section of the boundary is the neighborhood of point A_s is rectilinear, its contribution $O(\nu^{-1})$ to the asymptotics is the highest.

4. Let us consider bottom irregularities of specific form. Let h be a function of

the type (2.5). For an arbitrary function $h_3(u)$, $u = (\alpha x^2 + \beta y^2)^{1/2}$ the integral Π_1 is determined by numerical integration. In a number of cases it is defined in terms of elementary and special functions, for instance

$$h_3(u) = (1 - u^2 l^{-2})^{\mu} \quad (u \leqslant l), \quad h_3 = 0 \quad (u > l) \tag{4.1}$$
$$\Pi_1 = \pi 2^{\mu+1} (\alpha\beta)^{-1} l^2 \Gamma(\mu + 1) \cos(\gamma - \gamma_0) \xi^{-\mu-1} J_{\mu+1}(\xi)$$
$$h_3(u) = l^3 (l^2 + u^2)^{-3/2}, \quad \Pi_1 = 2\pi l^2 \cos(\gamma - \gamma_0) \exp(-\xi)$$
$$h_3(u) = \exp(-l^{-2} u^2), \quad \Pi_1 = \pi l^2 \cos(\gamma - \gamma_0) \exp(-{}^1/_4 \xi^2)$$
$$(\xi = m_2 l, \ u \geqslant 0, \ \mu \geqslant 0, \ l > 0)$$

where ($\Gamma(z)$ is the gamma function.

For h_3 of the form (4.1) the scattered wave amplitude is maximum when $\gamma = \gamma_0$. The ratio of wave amplitude at $\gamma = \gamma_0$ to that at $\gamma = \gamma_0 + \pi$ is the greater the greater are m_0 and l. When $h_3 \in C^{\infty}$, the scattered wave amplitude is zero along the rays $\gamma = \gamma_0 \pm \pi / 2$, however, if h_3 is not infinitely differentiable (the first expression for h_3 in (4.1)),other directions γ exist with weakly defined

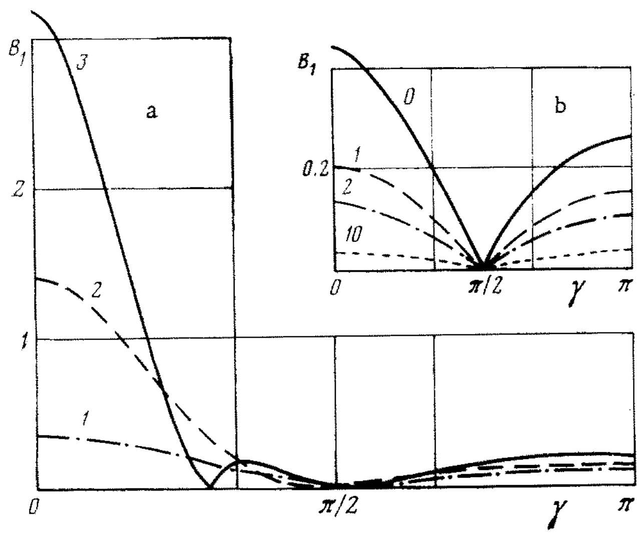


Fig. 2

diffraction effects, when m_0 and l are fairly large.

The distribution of scattered wave amplitude with respect to γ is defined by the parameter $B_1 = B \mid \Pi_1 \mid$. The dependence of B_1 on $\gamma \in [0, \pi]$

$$h_3 = \cos^s(\pi R / 2l) \quad (R \leqslant l), \ h_3 = 0 \quad (R > l) \tag{4.2}$$

is shown in Fig. 2 for the case of an axisymmetric irregularity of the bottom ($\gamma_0 = 0$). In diagram a of Fig. 2 the half-width l of the bottom irregularity ($r_0 = 1.5, s = 0$) is varied, and in diagram b it is the exponent s in formula (4.2) with ($r_0 = l = 1$). Values of parameters l and s are indicated at the respective curves.

The comparison of curves in Fig. 2 shows that the amplitude of the scattered wave generally diminishes as the width of the bottom irregularity decreases and its smoothness increases.

Let h be a function of the form (2.6) which corresponds to a nonaxisymmetric irregularity of the bottom. The scattered wave amplitude, nevertheless, weakly depends on the shape of the bottom irregularity when the incident wave is fairly long (see Sect. 3).

We determine functions $h_s(u)$, $s = 1, 2$ using formulas

$$h_s = 1 \quad (|u| \leqslant l_s), \quad h_s = 0 \quad (|u| > l_s) \tag{4.3}$$

$$h_s = \cos k_s u \quad (|u| \leqslant l_s), \quad h_s = 0 \quad (|u| > l_s), \quad k_s = \pi (2l_s)^{-1} \tag{4.4}$$

$$h_s = \exp(-3l_s^{-2}u^2) \tag{4.5}$$

When $m_0 \max\{l_1, l_2\} \geqslant 1$ the shape of irregularity affects in the case of such h the distant field of scattered waves. It manifests itself in the dependence of B_1 on γ_0 when $\gamma = \gamma_0 + \pi$ and in the impairment of symmetry of the scattered wave amplitude over the angle relative to the ray $\gamma = \gamma_0$.

The dependence of B_1 on γ for four wave incidence angles γ_0 and $r_0 = 0.1$, $l_1 = 0.1$, $l_2 = 10$ are shown in Fig. 3. Curves 1—3 relate to bottom irregularities for which functions h_s, $s = 1, 2$, are specified by formulas (4.3)—(4.5), respectively. This shows that the scattered wave has its highest amplitude in the neighborhood of the ray $\gamma = \gamma_0$. The value of $B_1|_{\gamma=\gamma_0} = B\,|F_1(0)|\,|F_2(0)|$ is independent of γ_0. As in the case of axisymmetric irregularity of the bottom, the improvement of the irregularity smoothness generally lowers the scattered wave amplitude.

5. The problem of surface wave diffraction over a small irregularity of the bottom of the form $h = h(x)$ is considered in a similar manner. Let us assume that the incident wave propagates at angle γ_0 to the x-axis. We set

$$\begin{aligned}
&\varphi = \operatorname{Re}\{(\Phi_0 + \Phi)\exp[i(n_0 y - \sigma t)]\} \\
&\zeta = \operatorname{Re}\{(A_0 \exp(im_0 x) + A)\exp[i(n_0 y - \sigma t)]\} \\
&\Phi_0 = -iA_0 g\sigma^{-1} \operatorname{ch} r_0 (z + H_0) \operatorname{ch}^{-1} r_0 H_0 \exp(im_0 x) \\
&r_0 = (m_0^2 + n_0^2)^{1/2}, \quad \sigma = (gr_0 \operatorname{th} r_0 H_0)^{1/2}
\end{aligned} \tag{5.1}$$

and introduce dimensionless variables using formulas of Sect. 1. For the determination of $\Phi(x, z)$ we obtain the boundary value problem

$$\begin{aligned}
&\Phi_{xx} + \Phi_{zz} - n_0^2\Phi = 0 \\
&\Phi_z - \sigma^2\Phi = 0 \ (z = 0), \ \Phi_z = h_x\Phi_{0x} - h\Phi_{0zz} \ (z = -1) \\
&\Phi_0 = -i \operatorname{ch} r_0 (z + 1) \operatorname{ch}^{-1} r_0 \exp(im_0 x), \quad \sigma = (r_0 \operatorname{th} r_0)^{1/2}
\end{aligned} \tag{5.2}$$

The scattered wave amplitude is calculated by formula $A = i\Phi(x, 0)$. The integral representation for A is derived from problem (5.2) using the Fourier transform with respect to x

$$A = \frac{1}{2\pi \operatorname{ch} r_0} \int_c (m_0 m + n_0^2) F(m - m_0) \times \tag{5.3}$$

$$[(r - \sigma^2 \operatorname{th} r) \operatorname{ch} r - \Delta \operatorname{sh} r] r^{-1} \Delta^{-1} \exp(imx)\, dm$$

where $(r = (m^2 + n_0^2)^{1/2},\ \Delta = r \operatorname{th} r - \sigma^2)$, c is the integration path in the complex m-plane that runs on the real axis bypassing poles $m = -m_0$ and $m = m_0$ along small semicircles above and below them, respectively, and $F(m)$ is the Fourier transform of function $h(x)$.

Integral (5.3) is similar to that considered in [10]. When $|x| \to \infty$, its asymptotics is of the form

$$A = A_\pm \exp(\pm i m_0 x) + O(e^{-\delta|x|}) \quad (x \to \pm\infty,\ \delta > 0) \tag{5.4}$$

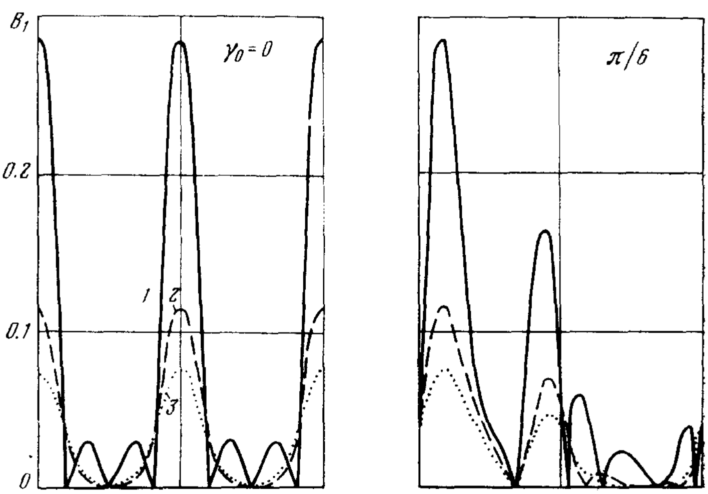
$$A_+ = i\Psi(r_0) F(0) \sec\gamma_0, \quad \Psi = r_0 [1 + \operatorname{sh}(2r_0)(2r_0)^{-1}]^{-1} \tag{5.5}$$

$$A_- = -i\Psi(r_0) F(-2r_0\cos\gamma_0)\cos 2\gamma_0 \sec\gamma_0 \tag{5.6}$$

Formulas (5.1) and (5.4) show that the incident wave generates a field of scattered surface waves which includes the reflected and passing waves whose amplitudes are of order ε and, also, the system of waves whose amplitudes exponentially decrease with increasing $|x|$ and which are localized in the neighborhood of the bottom irregularity. The latter waves propagate along the irregularity when $n_0 \neq 0$ and become standing waves when $n_0 = 0$. Thus an irregularity of the form $h = h(x)$ has wave guiding properties [11].

Formulas (5.4) show that the scattered wave amplitude is the same for incidence angles $\pm\gamma_0$ and is an increasing function of $\gamma_0 \in [0, \pi/2)$. Function $|A_+|$, as a function of r_0, has the unique maximum $\max_{r_0}|A_+| \approx 0.37\,|F(0)\sec\gamma_0|$ that is attainable for $r_0 \approx 1.2$ (Fig. 4).

The dependence of the reflected wave amplitude on r_0 and γ_0 is more complex. For small r_0 in conformity with (5.5) and (5.6), we have $|A_-| \approx |A_+|\,|\cos 2\gamma_0|$, hence $A_- = 0$ only when $\gamma_0 = \pi/4$. This case is represented in Fig. 4, where curves 1–3 correspond to functions $h(x)$ of the form (4.3)–(4.5), and $r_0 = 0.1$ and $l = 1$. If $F(\xi)$ is an oscillating function and r_0 fairly large, there exists a



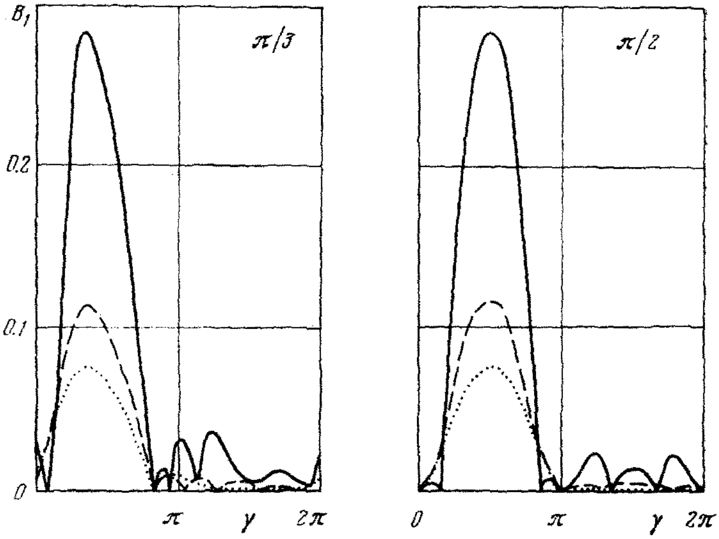


Fig. 3

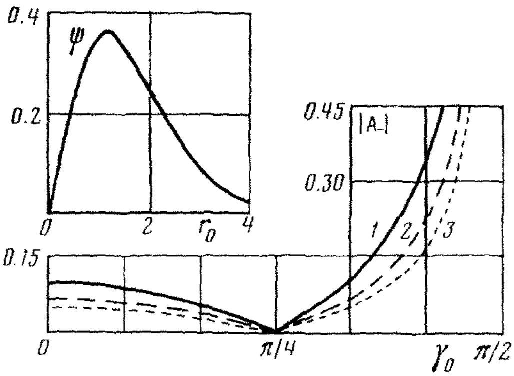


Fig. 4

set of incidence angles γ_0 for which no reflection wave is generated. An increase of r_0 and improved smoothness of the bottom irregularity generally result in a decrease of $|A_-|$ when $\gamma_0 \in [0, \pi/4)$.

The proposed approximate treatment of the problem of wave scatter over bottom irregularities of the form $h = h(x)$ is inapplicable when $\gamma_0 \to \pm \pi/2$.

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